A weighted surface-depth gradient method for the numerical integration of the 2D shallow water equations with topography

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A finite volume MUSCL scheme for the numerical integration of 2D shallow water equations is presented. In the framework of the SLIC scheme, the proposed weighted surface-depth gradient method (WSDGM) computes intercell water depths through a weighted average of DGM and SGM reconstructions, in which the weight function depends on the local Froude number. This combination makes the scheme capable of performing a robust tracking of wet/dry fronts and, together with an unsplit centered discretization of the bed slope source term, of maintaining the static condition on non-flat topographies (C-property). A correction of the numerical fluxes in the computational cells with water depth smaller than a fixed tolerance enables a drastic reduction of the mass error in the presence of wetting and drying fronts. The effectiveness and robustness of the proposed scheme are assessed by comparing numerical results with analytical and reference solutions of a set of test cases. Moreover, to show the capability of the numerical model on field-scale applications, the results of a dam-break scenario are presented.

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1. Introduction

Two-dimensional shallow water equations (SWE) are currently accepted to mathematically describe a wide variety of free surface flows under the effect of gravity, such as dam-break waves, propagation of flood waves in rivers, flood plain inundations, etc. Because the analytical solutions available in the literature concern only a few simple situations [24,27,29,25,22,18], the SWE need to be numerically solved in order to deal with more general applications.

The irregular topography of the regions subject to flooding can strongly affect the flow dynamics, giving rise to the formation of hydraulic jumps, shocks and reflections; for this reason, in the numerical simulation of such phenomena an efficient treatment of the bed slope source term is necessary to obtain accurate results both in the case of steady and unsteady flows.

In engineering applications the necessity to handle wetting and drying moving boundaries is a challenge that researchers are also tackling [4,5,13,10,6,7]. It is well known that small water depths near wet/dry interfaces can lead to numerical instabilities. The simple procedure of drying cells in which water depth is smaller than a fixed tolerance [31] avoids these unphysical oscillations, but it is not completely satisfying, because it induces a mass error that grows into not acceptable values in field-scale applications [11,7,23]. Another numerical difficulty is related to partially wet cells in which pressure and gravity forces are not exactly balanced in static conditions [15,10].

In recent years a large amount of research has dealt with the application of finite volume Godunov-type methods for the numerical solution of SWE with source terms [30,8,14,9,33,16,19,34,26]. In the fractional step method [31,20] the complete equations are split into a homogeneous problem and an ODE system. Although simple, the procedure performs poorly if applied to SWE with geometric source terms, especially near the steady state [31,20]. A more complex approach, which gives more satisfactory results, concerns the upwinding of the source terms, in a manner similar to that adopted for the construction of numerical fluxes for solving homogeneous conservation laws. Bermúdez and Vázquez-Cendón [8] applied this treatment to first order Roe’s scheme and defined the fundamental notion of C-property, that is the capability of replicating the exact solution for the stationary flow problem. This upwinding approach was extended to a wide range of problems by García-Navarro and Vázquez-Cendón [14], Bermúdez et al. [9] and Vázquez-Cendón [33]. Later Hubbard and García-Navarro [16] generalized this technique to a finite volume high order TVD version of Roe’s scheme and to arbitrary polygonal meshes. In [19] LeVeque proposed the quasi-steady state wave propagation algorithm in which, avoiding any splitting, the source term is incorporated into the complete equations and an approximate Riemann solver is used. This approach allows a drastic reduction of the mass error in the presence of wetting and drying fronts. However, the computational cost is higher than the first-order solution with a simple upwinding approach. In the present paper we present a method to improve current approximations of the source terms with the aim of reducing the mass error in situations of both wetting and drying fronts.
the wave propagation algorithm; a Riemann problem is introduced at the center of each cell whose flux difference exactly cancels the source term. According to the author, this method is effective when the solution is near the steady state, but it presents difficulties in the case of transcritical steady flows with shocks. In Zhou et al. [34] the surface gradient method was introduced. In order to evaluate more accurate numerical fluxes in the presence of a non-flat topography, the intercell water depth was computed starting from the MUSCL reconstruction of the water surface level. Together with a centered discretization of the bottom slope source term, the scheme is capable of maintaining the static condition on a Cartesian grid. This approach is very attractive for its simplicity, but, near wet/dry fronts on uneven topographies, the SGM reconstruction can lead to very small water depths (even negative) that need to be modified in order to avoid unphysical results. A more robust behavior in the front tracking can be obtained with the conventional DGM (depth gradient method) approach which evaluates intercell water depths starting from the extrapolation of the same conserved variable; on the other hand, this method does not maintain the static condition if a centered discretization is used for the bed slope source term. In the recent work of Valiani and Begnudelli [32] the bed slope source term is expressed as the divergence of a suitable matrix related to the static force due to the bottom slope; moreover, a SGM variable reconstruction is performed to compute water depth at the cell boundaries. This technique is simple and allows the preservation of the condition of quiescent water on a totally wet domain on even though the grid is irregular.

In this paper a weighted surface depth gradient method (WSDGM) is proposed with the aim of combining the good capabilities of SGM and DGM approaches and avoiding their drawbacks. In the framework of the SLIC scheme [31], the proposed algorithm computes water depths at the cell boundaries through a weighted average of the extrapolated values deriving from DGM and SGM reconstructions: in this way the scheme is capable of maintaining the static condition on non-flat topographies and performing a robust tracking of wet/dry fronts.

In cells in which the computed water depth is lower than a small threshold, a flux correction is applied in order to drastically reduce the mass error. The effectiveness and robustness of the proposed scheme on non-horizontal beds and in the presence of wet/dry fronts is checked on a set of test cases for which an analytical solution is available or a highly accurate numerical solution can be derived.

Finally, the simulation of a real case study concerning the hypothetical collapse of the dam on the Parma river (Northern Italy) confirms the applicability of WSDGM to field-scale problems on irregular topographies with wetting and drying.

2. Numerical model

Under the usual De St. Venant hypotheses of small bottom slopes and small streamline curvatures, the integral form of SWE reads [31]:

$$\frac{\partial}{\partial t} \int_A U \, dA + \int_C H \cdot n \, dC = \int_A (S_0 + S_1) \, dA. \tag{1}$$

In (1) \( A \) is the area of the integration element, \( C \) the element boundary, \( n \) the outward unit vector normal to \( C \), \( U \) the vector of the conserved variables and \( H = \text{[F,G]} \) the tensor of fluxes in the \( x \) and \( y \) directions:

$$U = \begin{bmatrix} h \\ uh \\ vh \\ u^2 h + \frac{1}{2} gh^2 \\ uhv \\ uvh \\ \sqrt{u^2 h + \frac{1}{2} gh^2} \end{bmatrix}, \quad F = \begin{bmatrix} uh \\ u^2 h + \frac{1}{2} gh^2 \\ uhv \\ \sqrt{u^2 h + \frac{1}{2} gh^2} \end{bmatrix}, \quad G = \begin{bmatrix} vh \\ uvh \\ \sqrt{v^2 h + \frac{1}{2} gh^2} \end{bmatrix}, \tag{2}$$

where \( h \) is the flow depth, \( u \) and \( v \) are the velocity components in the \( x \) and \( y \) directions and \( g \) is the acceleration due to gravity. The bed and friction slope source terms \( S_0 \) and \( S_1 \) are expressed according to the following relations:

$$S_0 = \left[ 0, \ -gh \frac{\partial h}{\partial x} \ - gh \frac{\partial h}{\partial y} \right]^T,$$

$$S_1 = \left[ 0, \ -gh \frac{\partial u}{\partial x} \ + \frac{\partial h}{\partial x} \ + \frac{\partial h}{\partial y} \ - gh \frac{\partial v}{\partial x} \ + \frac{\partial h}{\partial x} \ + \frac{\partial h}{\partial y} \right]^T, \tag{3}$$

in which \( z \) is the bed level with respect to a horizontal reference and \( n_t \) is the roughness coefficient of the Manning equation.

Among the numerical techniques for the integration of (1), the finite volume method allows the computation of the variation \( \Delta U \) over the time interval \( \Delta t \) as a function of the numerical fluxes exchanged through the boundaries and of the forcing source terms. If applied to a grid composed of polygonal elements with \( N \) sides, the variation \( \Delta U \) takes the form [30,20]:

$$\Delta U = -\frac{\Delta t}{A} \sum_{k=1}^N (f_k - n_k) \Delta k + \frac{\Delta t}{A} \int_A (S_0 + S_1) \, dA. \tag{4}$$

where \( n_k \) is the outward unit vector normal to the \( k \)-side of the cell of length \( \Delta k \). If the grid is composed of Cartesian cells of area \( \Delta x \times \Delta y \) and the first order Godunov splitting of the source terms is applied [30], the explicit updating algorithm for the discretized variable \( U_{ij} \) becomes:

$$\begin{cases} U_{ij}^n = U_{ij}^{n-1} + \frac{\Delta t}{C_3} \left( F_{i+\frac{1}{2},j} - F_{i-\frac{1}{2},j} \right) - \frac{\Delta t}{C_3} \left( S_{i+\frac{1}{2},j} - S_{i-\frac{1}{2},j} \right) \\ U_{ij}^{n+1} = U_{ij}^{n-1} + \frac{\Delta t}{C_3} \left( S_{ij+\frac{1}{2}} + S_{ij-\frac{1}{2}} \right) \end{cases} \tag{5}$$

In (5) the superscript \( n \) denotes the current time level, while \( F_{i+1/2,j} \) and \( G_{i,j+1/2} \) are the numerical fluxes exchanged through the boundaries between neighboring cells in the \( x \) and \( y \) directions respectively.

The numerical fluxes can be computed using several methodologies. In previous works [2,3] the writers adopted the well-known Slope Limiter Centred scheme (SLIC) [31] belonging to the MUSCL family and to the depth gradient method (DGM) class. The scheme consists of the following steps.

(a1) A set of boundary extrapolated variables is evaluated at the interfaces of each computational cell to achieve second order of accuracy in space. For direction \( x \) it reads:

$$U_{i+\frac{1}{2},j}^n = U_{ij}^n + \frac{1}{2} \Phi^+_{i+\frac{1}{2},j} \left( U_{ij}^n - U_{i-1,j}^n \right), \tag{6}$$

$$U_{i-\frac{1}{2},j}^n = U_{ij}^n - \frac{1}{2} \Phi^-_{i-\frac{1}{2},j} \left( U_{ij}^n - U_{i+1,j}^n \right).$$

In (6), the limiter diagonal matrices \( \Phi \) [28] enforce a TVD constraint on the slope of the boundary extrapolated variables in order to avoid spurious oscillations. These limiters are a function of the ratios \( \Phi^\ast \) of the consecutive variations:

$$\Phi^+ = \Phi \left( r_{i+\frac{1}{2},j} \right), \quad \Phi^- = \Phi \left( r_{i-\frac{1}{2},j} \right), \tag{7}$$

with vectors \( r^\ast \) defined as:

$$r_{i+\frac{1}{2},j} = U_{ij}^n - U_{i-1,j}^n, \quad r_{i-\frac{1}{2},j} = U_{ij}^n - U_{i+1,j}^n. \tag{8}$$

Among several slope limiter functions proposed in the literature [20], one of the following may be adopted:
According to the DGM technique, it is labeled as:
\[
h_{i+1,j}^h = h_{i,j}^h + \frac{1}{2} \phi_{i+1/2}^j (h_{i+1,j}^n - h_{i,j}^n),
\]
\[
h_{i,j}^h = h_{i-1/2,j}^h + \frac{1}{2} \phi_{i-1/2}^j (h_{i,j+1}^n - h_{i,j}^n).
\]

(b1) The boundary extrapolated variables are evolved over \(\Delta t/2\) in order to achieve second order of accuracy in time:
\[
\overline{U}_{i+1/2,j}^t = U_{i+1/2}^t - \frac{\Delta t}{2 A\lambda} \left[ G(U_{i+1/2,j}^t) - G(U_{i+1/2}^t) \right] \\
- \frac{\Delta t}{2 A\lambda} \left[ G(U_{i+1/2,j}^t) - G(U_{i+1/2}^t) \right],
\]
\[
\overline{U}_{i,j+1/2}^t = U_{i,j+1/2}^t - \frac{\Delta t}{2 A\lambda} \left[ G(U_{i,j+1/2}^t) - G(U_{i,j+1/2}^t) \right] \\
- \frac{\Delta t}{2 A\lambda} \left[ G(U_{i,j+1/2}^t) - G(U_{i,j+1/2}^t) \right].
\]

(c1) The numerical intercell fluxes are evaluated according to the first O'Rder Cntented (FORCE) method [31]:
\[
F_{i+1/2,j}^{\text{force}} = F_{i+1/2,j}^{\text{force}} \left( \overline{U}_{i+1/2,j}^t, \overline{U}_{i,j+1/2}^t \right) \\
= \frac{1}{2} \left[ F_{i+1/2,j} (\overline{U}_{i+1/2,j}^t + \overline{U}_{i,j+1/2}^t) + F_{i,j+1/2} (\overline{U}_{i+1/2,j}^t + \overline{U}_{i,j+1/2}^t) \right],
\]
\[
G_{i+1/2,j}^{\text{force}} = G_{i+1/2,j}^{\text{force}} \left( \overline{U}_{i+1/2,j}^t, \overline{U}_{i,j+1/2}^t \right) \\
= \frac{1}{2} \left[ G_{i+1/2,j} (\overline{U}_{i+1/2,j}^t + \overline{U}_{i,j+1/2}^t) + G_{i,j+1/2} (\overline{U}_{i+1/2,j}^t + \overline{U}_{i,j+1/2}^t) \right].
\]

The illustrated scheme is robust and stable when high water level gradients occur together with small water depth gradients, as shown in the case of a supercritical steady flow over a bump, presented in Section 3.1. Introducing a threshold value \(h\), to avoid instabilities in the presence of very small depths [31], the scheme is also capable of tracking the motion of wetting and drying fronts. However, if a centered discretization is used for the bed slope term, this approach can not replicate the exact solution of the static flow problem \(\eta(x,y,t) = h(x,y,t) + z(x,y) = \eta_i, \ u(x,y,t) = 0, v(x,y,t) = 0\), i.e. does not satisfy the exact C-property [8].

This C-property can be instead satisfied on a Cartesian grid if the Surface Gradient Method is applied in the data reconstruction step [34].

If this method is incorporated into the SLIC scheme, numerical fluxes are computed by the following steps.

(a2) The evaluation of water depths at cell interfaces is obtained from the reconstruction of the intercell water level \(\eta = h + z\) [34]:
\[
\eta_{i+1/2,j} = \eta_{i,j} + \frac{1}{2} \phi_{i+1/2}^j \left( \eta_{i+1/2,j} - \eta_{i,j} \right) \\
+ \frac{\Delta t}{2 C_{16}/C_{17}} \left( \eta_{i+1/2,j}^n - \eta_{i,j}^n \right),
\]
\[
\eta_{i,j+1/2} = \eta_{i,j} - \frac{1}{2} \phi_{i,j+1/2} \left( \eta_{i+1/2,j} - \eta_{i,j} \right) \\
+ \frac{\Delta t}{2 C_{16}/C_{17}} \left( \eta_{i+1/2,j}^n - \eta_{i,j}^n \right),
\]
\[
h_{i+1/2,j} = h_{i,j} + \frac{1}{2} \phi_{i+1/2}^j \left( h_{i+1/2,j}^n - h_{i,j}^n \right),
\]
\[
h_{i,j+1/2} = h_{i,j} - \frac{1}{2} \phi_{i,j+1/2} \left( h_{i+1/2,j}^n - h_{i,j}^n \right).
\]

Surface Gradient Method is applied in the data reconstruction step (a2). The evaluation of water depths at cell interfaces is obtained from the reconstruction of the intercell water level \(\eta = h + z\) [34]:

(b2) In order to satisfy the C-property when SGM is adopted, the evolution of the conserved variables over \(\Delta t/2\) must include the contribution due to gravity:
\[
\overline{U}_{i+1/2,j}^t = U_{i+1/2}^t - \frac{\Delta t}{2 A\lambda} \left[ G(U_{i+1/2,j}^t) - G(U_{i+1/2}^t) \right] + \frac{\Delta t}{2 S_{0i,j}} \\
- \frac{\Delta t}{2 A\lambda} \left[ G(U_{i+1/2,j}^t) - G(U_{i+1/2}^t) \right] + \frac{\Delta t}{2 S_{0i,j}} \\
- \frac{\Delta t}{2 A\lambda} \left[ G(U_{i+1/2,j}^t) - G(U_{i+1/2}^t) \right] + \frac{\Delta t}{2 S_{0i,j}} \\
- \frac{\Delta t}{2 A\lambda} \left[ G(U_{i+1/2,j}^t) - G(U_{i+1/2}^t) \right] + \frac{\Delta t}{2 S_{0i,j}}.
\]

(c2) Numerical fluxes \(F_{i+1/2,j}, G_{i+1/2,j}\) are evaluated via the FORCE method described at point (c1).

After the computation of numerical fluxes, the solution is updated replacing (5) by the following algorithm:
\[
\begin{align*}
U_{i,j}^{t+1} &= U_{i,j}^{t} - \Delta t \left( S_{i,j} + \eta_{i+1/2,j}^{t+1} \right), \\
G_{i+1/2,j}^{t+1} &= G_{i+1/2,j}^{t+1} - \Delta t \left( S_{i,j} + \eta_{i+1/2,j}^{t+1} \right),
\end{align*}
\]
in which \(S_{i,j}\) is discretized with the centered approximation (17) as a function of the evolved variables \((\overline{h}_{i+1/2,j}^{t+1}, \overline{h}_{i+1/2,j}^{t+1})\). In (18) the unsplitting of the bed slope source term guarantees the exact conservation of the static condition on a completely wet irregular topography.

The SGM is equivalent to the DGM in the case of a flat bottom, and entails an equivalent computational effort [34]. SGM reconstructions are preferable for those applications in which small water level gradients occur in the presence of high water depth gradients, as, for example, in the case of a subcritical steady flow over a bump, as reported in Section 3.1. The application of SGM to field-scale cases often leads to unsatisfactory results in the treatment
of the wetting and drying fronts, especially on highly irregular topographies. In fact, near wet/dry interfaces on a non-flat bottom, the SGM reconstruction can give rise to very small water depths (even negative) and, as a consequence, to unphysical results.

In order to overcome the limitations of SGM and DGM reconstructions and, at the same time, retain their good capabilities, a weighted surface-depth gradient method was implemented according to the following computational steps.

(a3) In the data reconstruction step, the water depth at cell interfaces is estimated through a weighted average of the boundary extrapolated values deriving from MUSCL DGM and SGM extrapolations:

\[ h_{i,j}^L = \vartheta h_{i,j}^{\text{DGM}} + (1 - \vartheta) h_{i,j}^{\text{SGM}}; \]

\[ h_{i,j}^R = \vartheta h_{i,j}^{\text{DGM}} + (1 - \vartheta) h_{i,j}^{\text{SGM}}; \]

in which \( \vartheta \) is a weighting parameter.

The form of \( \vartheta \) should allow a smooth transition between a fully SGM extrapolation where water is at rest (Fr = 0) and an essentially DGM extrapolation at wet/dry moving fronts. Both previous requirements can be satisfied by adopting the Froude number as the control parameter.

Although any function with the features mentioned above can be used, WSDGM adopts a simple trigonometric expression for the weighting function:

\[ \vartheta_{ij} = \left\{ \begin{array}{ll} \frac{1}{2} \left[ 1 - \cos \left( \frac{\pi F_{\text{lim}}}{F_{\text{lim}}} \right) \right] & \text{if } 0 < F_{\text{rij}} \leq F_{\text{lim}} \\ 1 & \text{if } F_{\text{rij}} > F_{\text{lim}}, \end{array} \right. \]

where \( F_{\text{lim}} \) is an upper limit beyond which a pure DGM reconstruction is performed.

Since a shock wave cannot be adjacent to a region of dry bed [31], water depths must tend gradually to zero and Fr to infinite at wet/dry interfaces; however, in a finite volume framework, Fr values remain finite also at the shoreline: the choice of \( F_{\text{lim}} \) needs to guarantee an essentially DGM-type behavior near wet/dry moving fronts.

In order to rigorously satisfy the C-property and to prevent the well-known instabilities [4,5,13,6] that arise at the shoreline when a fixed mesh is adopted, a special treatment is performed at wet-dry interfaces. If the water surface in the \((i,j)\)-cell is lower than the bed elevation of the adjacent dry cell, the intercell bed elevation is set at the level of the extrapolated water surface and, as a consequence, the reconstructed water depth is zero [15,1]. The centered estimate of the intercell bed elevation is restored when the shoreline no longer involves the cell.

(b3) The evolution of the extrapolated variables includes the forcing effect of the bed slope source term as in Step (b2).

(c3) Numerical fluxes are computed according to Step (c1).

The solution is updated applying algorithm (18). In problems involving friction the source term \( S_h \) is discretized applying a second order semi-implicit trapezoidal method [12].

The occurrence of very small water depths in numerical simulations can lead to instabilities, such as negative water depths and unphysical velocities; moreover, in problems with friction, the structure of the Manning equation is such that, when water depth tends to zero, the bed resistance tends to infinite. In order to avoid these difficulties, computed waterdepths lower than a fixed tolerance are usually set at zero. This common procedure leads to a loss of mass when the updated water depth is positive but smaller than \( h_n \), and to a gain when it is negative. In real case studies in which fronts can be very uneven due to the strong bottom irregularities, the mass error can grow into unacceptable values.

In order to drastically reduce this mass error, when the water depth \( h_{ij} \) deriving from the first step of (18) is lower than \( h_n \), in WSDGM a flux correction is performed to obtain \( h_{ij} = 0 \).

Defining:

\[ C_{ij} = \frac{h_{ij} \Delta y}{\Delta t \left[ \left( c_1 \cdot F_{ij}^1 + c_2 \cdot F_{ij}^2 \right) \Delta y + \left( c_3 \cdot G_{ij}^1 + c_4 \cdot G_{ij}^2 \right) \Delta x \right]}; \]

where \( F^1 \) and \( G^1 \) are the first component of the numerical fluxes in \( x \) and \( y \) directions and \( c_k \) \((k = 1, 2, 3, 4)\) are integer coefficients equal to 0 or 1, three different cases can be distinguished:

- \((0 < h_{ij} \leq h_n) \cap (h_{ij}^y = 0)\), as occurs, for example, at the wetting front, when the cell, initially dry, is wetted with a water depth smaller than \( h_n \). If \( F^1 \cdot n < 0 \) or, similarly, \( G^1 \cdot n < 0 \) (flow entering the \((i,j)\)-cell), the correspondent coefficient \( c_k \) is set at 1, otherwise at 0. If \( c_k = 1 \) all the components of the numerical flux vector are reduced by the same factor \( x_{ij} = (1 - C_{ij}) \), otherwise the flux vector remains unchanged;

- \((0 < h_{ij} \leq h_n) \cap (h_{ij}^y > h_n)\), as occurs, for example, at the drying front, when the cell, initially wet, is not completely dried. If \( F^1 \cdot n > 0 \) or, similarly, \( G^1 \cdot n > 0 \) (flow leaving the \((i,j)\)-cell), the correspondent coefficient \( c_k \) is set at 1, otherwise at 0. If \( c_k = 1 \) all the components of the numerical flux vector are increased by the same factor \( x_{ij} = (1 + C_{ij}) \), otherwise the flux vector remains unchanged;

- \((h_{ij} \leq 0)\), as occurs, for example, at a drying front, when the cell is overdrawn. If \( F^1 \cdot n > 0 \) or, similarly, \( G^1 \cdot n > 0 \) (flow leaving the \((i,j)\)-cell), the correspondent coefficient \( c_k \) is set at 1, otherwise at 0. If \( c_k = 1 \) all the components of the numerical flux vector are reduced by the same factor \( x_{ij} = (1 - C_{ij}) \), otherwise the flux vector remains unchanged.

After the flux correction, the first step of (18) is recomputed; since this procedure can lead to \( h_{ij} < h_n \) in the adjacent cells, it is iteratively performed: in this way the mass error is drastically reduced, without significantly increasing the computational effort, because the cells involved by the algorithm are always a small percentage of the total number of elements at wet/dry interfaces.

In order to compute numerical fluxes at the cell sides placed on the boundary of the computational domain, boundary conditions for the conserved variables \( h, uh, vh \) and the bottom elevation \( z \) are imposed on ghost cells adjacent to the domain [20], according to the different character of the flow.

The proposed WSDGM is globally second order accurate in space thanks to the MUSCL variable reconstruction [31], but first order accurate in time due to the splitting of the friction source term (Eq. (18)) [20]; for frictionless problems, the second order of accuracy is achieved also in time.

The time step \( \Delta t \) is controlled by means of the 2D CFL stability criterion \( 0 < C_r + C_t \leq 1 \) [30], where \( C_r \) and \( C_t \) are the Courant numbers in \( x \) and \( y \) directions respectively. This assumption leads to:

\[ \Delta t = \min \left( \frac{\max \left( |u_{ij}| + \sqrt{c h_{ij}} \Delta x / \Delta y, |v_{ij}| + \sqrt{c h_{ij}} \Delta y / \Delta x \right)}{C_r}, \frac{\max \left( |u_{ij}| + \sqrt{c h_{ij}} \Delta x / \Delta y, |v_{ij}| + \sqrt{c h_{ij}} \Delta y / \Delta x \right)}{C_t} \right)^{-1}, \]

where \( 0 < C_r \leq 1 \).

### 3. Test cases with reference solution

In this section the robustness and effectiveness of the proposed scheme are assessed applying WSDGM to non-flat topography reference test cases without friction. The dynamics of wetting and drying and the occurrence of complex structures in the flow field, such as moving shocks and reflections, mean that the tests are severe.
Among the possible options proposed in the literature, the Van Leer limiter function was adopted.

3.1. 1D steady flows over a steep bump

These tests concern 1D steady flows in a \([-10 \text{ m} \leq x \leq 10 \text{ m}]\) frictionless channel. The bottom profile, characterized by the presence of a steep bump, is described by the following equation:

\[
(z(x) = \begin{cases} 
    b_i \left(1 - \frac{x^2}{b_i^2}\right) & -2 \text{ m} \leq x \leq 2 \text{ m} \\
    0 & \text{otherwise}
\end{cases}
\] (23)

where the height of the bump \(b_i\) is set at 0.8 [21] instead of the usual value 0.2 (e.g. [34,32,7]), in order to make the test more severe. The computational domain is discretized with 0.1 m grid spacing.

According to the conditions imposed at the boundaries (Table 1), the water can be at rest (Test A, with initial condition \(q(x) = 1 \text{ m}\)) or the flow can be transcritical with (Test B) or without (Test C) the occurrence of a hydraulic jump, supercritical (Test D) or subcritical (Test E).

The five cases were simulated by means of WSDGM with different values of \(Fr_{\text{lim}}\) ranging from \(Fr_{\text{lim}} = 0\) (pure DGM reconstruction) to \(Fr_{\text{lim}} \rightarrow \infty\) (pure SGM reconstruction).

Table 2 compares the \(L_1\) errors in \(h\) and \(q\) as a function of \(Fr_{\text{lim}}\), which are defined as:

\[
L_1(h) = \frac{1}{N} \sum_{i=1}^{N} |h_{\text{num}} - h_{\text{ref}}|, \\
L_1(q) = \frac{1}{N} \sum_{i=1}^{N} |q_{\text{num}} - q_{\text{ref}}|.
\] (24)

where \(N\) is the total number of computational cells, \(h_{\text{ref}}\) is the reference solution derived from the energy equation (together with the momentum principle if a hydraulic jump is present) and \(q_{\text{ref}}\) is the constant value of the specific discharge.

Even though in Test E a pure SGM reconstruction (\(Fr_{\text{lim}} \rightarrow \infty\)) gives a slightly better result, on average the lowest \(L_1\) errors are obtained for values of \(Fr_{\text{lim}}\) in the interval [1,2]. Since in this range the norms are almost the same, the value \(Fr_{\text{lim}} = 2\) is chosen to give insight into the results. Fig. 1 compares, in the subdomain \([-3 \text{ m} \leq x \leq 3 \text{ m}]\), reference water level and unit discharge profiles with those computed assuming \(Fr_{\text{lim}} = 0\), \(Fr_{\text{lim}} = 2\) and \(Fr_{\text{lim}} \rightarrow \infty\).

As shown by numerical profiles and errors in \(L_1\) norms for Test A, the pure DGM reconstruction (\(Fr_{\text{lim}} = 0\)) does not satisfy the C-property.

### Table 1: Boundary conditions and \(Fr\) range for 1D steady flows over a bump

<table>
<thead>
<tr>
<th>Test</th>
<th>Upstream boundary conditions</th>
<th>Downstream boundary conditions</th>
<th>(Fr) range</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>(q = 0.0 \text{ m/s}^2)</td>
<td>(h = 1.0 \text{ m})</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>(q = 0.4 \text{ m/s}^2)</td>
<td>(h = 0.75 \text{ m})</td>
<td>0.10–4.00</td>
</tr>
<tr>
<td>C</td>
<td>(q = 0.4 \text{ m/s}^2)</td>
<td>Transmissive</td>
<td>0.10–0.03</td>
</tr>
<tr>
<td>D</td>
<td>(q = 1.5 \text{ m/s}^2), (h = 0.25 \text{ m})</td>
<td>Transmissive</td>
<td>2.31–3.83</td>
</tr>
<tr>
<td>E</td>
<td>(q = 1.0 \text{ m/s}^2), (h = 1.70 \text{ m})</td>
<td>0.14–0.41</td>
<td></td>
</tr>
</tbody>
</table>

### Table 2: \(L_1\) error norms for \(h\) and \(q\) as function of \(Fr_{\text{lim}}\)

<table>
<thead>
<tr>
<th>(Fr_{\text{lim}})</th>
<th>Test A</th>
<th>Test B</th>
<th>Test C</th>
<th>Test D</th>
<th>Test E</th>
</tr>
</thead>
<tbody>
<tr>
<td>(L_1(h))</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>2.0E-3</td>
<td>3.2E-3</td>
<td>3.9E-3</td>
<td>4.4E-4</td>
<td>6.2E-4</td>
</tr>
<tr>
<td>0.2</td>
<td>0.0E+0</td>
<td>3.2E-3</td>
<td>3.7E-3</td>
<td>4.4E-4</td>
<td>4.6E-4</td>
</tr>
<tr>
<td>1</td>
<td>0.0E+0</td>
<td>3.1E-3</td>
<td>3.5E-3</td>
<td>4.3E-4</td>
<td>1.9E-4</td>
</tr>
<tr>
<td>2</td>
<td>0.0E+0</td>
<td>3.2E-3</td>
<td>3.5E-3</td>
<td>6.0E-4</td>
<td>1.6E-4</td>
</tr>
<tr>
<td>5</td>
<td>0.0E+0</td>
<td>3.3E-3</td>
<td>3.7E-3</td>
<td>6.5E-4</td>
<td>1.8E-4</td>
</tr>
<tr>
<td>20</td>
<td>0.0E+0</td>
<td>3.5E-3</td>
<td>4.8E-3</td>
<td>6.2E-3</td>
<td>5.4E-3</td>
</tr>
<tr>
<td>(\rightarrow \infty)</td>
<td>0.0E+0</td>
<td>4.0E-3</td>
<td>5.3E-3</td>
<td>1.1E-2</td>
<td>9.8E-3</td>
</tr>
</tbody>
</table>

In the subcritical regions of Tests B, C, E, the results obtained adopting \(Fr_{\text{lim}} = 0\) show a dip in the water elevation upstream and downstream of the bump, where the bottom profile is not smooth.

If the flow is supercritical downstream from the bump (Tests C and D), the numerical scheme with \(Fr_{\text{lim}} \rightarrow \infty\) badly converges toward the steady solution, as shown by \(L_1\) norms and oscillations in water elevation and unit discharge profiles. It was verified that tests analogous to B, C and D, characterized by boundary conditions giving rise to a thinner nappe over the last part of the bump, cannot be carried out if a pure SGM reconstruction is performed (e.g. Test B with \(q = 0.25 \text{ m/s}^2\) and \(h = 0.60 \text{ m}\)).

In Test B a deviation of the computed discharges from the reference solution occurs close to the hydraulic jump; a similar behavior can be found in [34,32] even though \(b_i = 0.2 \text{ m}\). This deviation is the main source of error and gives rise to comparable \(L_1\) norms for all the examined values of \(Fr_{\text{lim}}\).

When \(Fr_{\text{lim}} = 0\) and the flow is completely subcritical (Tests A and E) the bottom discontinuities induce deviations in \(q\) at the beginning and end of the bump; the same behavior occurs in Test D for \(Fr_{\text{lim}} \rightarrow \infty\).

On the whole, the results obtained for these test cases prove that WSDGM with \(Fr_{\text{lim}} = 2\) performs better than the schemes based on pure DGM or SGM reconstructions. For this reason, the parameter \(Fr_{\text{lim}}\) was set at 2 also in the following tests.

In order to numerically prove that WSDGM is second order accurate, a convergence analysis on the grid size was performed with reference to Test C. Four different grid spacings were used, starting from \(\Delta x = 0.2 \text{ m}\), and successively halving the value to obtain the remaining grids. Fig. 2 shows that the \(L_1\) norms in \(h\), \(q\) and energy per unit mass \(E\) vanish like \(O(\Delta x^2)\) when the grid is progressively refined.

### 3.2. Dam-break in a frictionless sloping channel

In this subsection, the rapidly varying 1D flow induced by a dam-break in a frictionless sloping channel is investigated (Fig. 3). Among the tests for which an analytical solution of wet/dry fronts is available [24,29], this is one of the most strongly influenced by the threshold value \(h_i\), because the water body lengthens indefinitely, becoming thinner and thinner. For this reason, the dam-break in a sloping channel was chosen as the reference test for the evaluation of the threshold effects on the front tracking.

Following Hunt [17] and introducing the set of dimensionless variables:

\[
\frac{x}{h_0} = \frac{h}{h_0} = \frac{t}{\sqrt{\frac{h_0}{gh_0}}} = \frac{u}{\sqrt{gh_0}}, \\
\frac{c}{\sqrt{gh_0}} = \sqrt{\frac{h}{h_0}}.
\] (25)

where \(h_0\) is the water depth behind the dam, \(S_0\) is the bed slope and \(c\) is the speed of small amplitude waves in still water, it is possible...
to rewrite the 1D De St. Venant equations in the dimensionless characteristic form:

\[
\begin{align*}
\frac{d}{dt} x_t &= u \pm \hat{c} \\
\frac{d}{dt} (u(x_t; t) \pm 2c(x_t; t) - t) &= 0.
\end{align*}
\] (26)

From (26) the dimensionless expression of the drying and wetting front \(x_{dry}, x_{wet}\) can be derived [17]:

\[
\begin{align*}
x_{dry} &= \frac{1}{2} (\ell - 2)^2 - 1 \quad (\ell \geq 2), \\
x_{wet} &= \frac{1}{2} (\ell + 2)^2 - 2.
\end{align*}
\] (27) (28)

In the numerical simulation a \([-10 \text{ m} \leq x \leq 150 \text{ m}]\) channel of slope \(S_0 = 0.1\) was discretized with a mesh of size 0.1 m. The dam section was placed at \(x = 0\) and the maximum water depth \(h_0\) was set at 1 m.

\[\text{Fig. 1. 1D steady flows over a steep bump: comparison between reference solutions and numerical results. In the shaded region } Fr > 2.\]
The value of the dimensionless threshold shown in Fig. 4. The wetting front is not particularly sensitive to the value of the dimensionless threshold \( h = h/h_0 \), but the drying process is more so. In particular, the numerical drying fronts obtained imposing \( h = 10^{-5} \) and \( h = 10^{-4} \) are respectively behind and ahead the analytical solution. The drying front is obviously influenced by the grid dimension too. Table 3 compares the \( L_2 \) error norms of the drying front position as a function of \( \Delta x \), which are defined as:

\[
L_2(x_{dny}) = \sqrt{\frac{1}{N \Delta x} \sum_{i=1}^{N} \left( x_{dny,i} - x_{dny,i}^{\text{num}} \right)^2},
\]

with \( N \) the number of computational time steps in the dimensionless interval \( 2 \leq T \leq 5 \). Despite the halving and the doubling of the grid size, it is confirmed that the analytical drying front lies between the numerical fronts obtained assuming \( h \) equal to \( 10^{-5} \) and \( 10^{-4} \). Thus, it can be stated that a value of \( h \) lying in the interval \( [10^{-5}, 10^{-4}] \) is suitable to achieve a satisfactory tracking of wet/dry interfaces.

### 3.3.2 2D periodic motions in a parabolic basin

The capability of the proposed method of providing accurate results in the presence of 2D wetting and drying moving boundaries on non-flat topographies was verified comparing numerical results with two exact solutions given by Thacker [29], which concern the oscillation of a water volume in a frictionless paraboloid basin of equation:

\[
z = z_0 \left( 1 - \frac{x^2 + y^2}{L^2} \right).
\]

In (30) the depth function \( z \) is positive below the equilibrium level (Fig. 5a and Fig. 7a), \( z_0 \) is the depth of the vertex of the paraboloid and \( L \) is the radius at \( z = 0 \).

In the following sections, two cases will be considered, corresponding to particular choices for initial values. In the first (Section 3.3.1) the water body rotates in the basin, maintaining its free surface planar and the velocity field uniform, while in the second (Section 3.3.2) the water surface is a parabola of revolution which expands and contracts periodically.

#### 3.3.1 Planar water surface

In this case the moving shoreline is a circle of radius \( L \) whose center \( C \) describes a circle of radius \( \xi \) (Fig. 5a). The equations of this motion are given by [29]:

\[
\begin{align*}
\eta(x,t) &= \frac{\xi^2}{2} \left[ \frac{\xi}{\xi \cos \omega t - \xi \sin \omega t} \right]
\end{align*}
\]

\[
\begin{align*}
u(x,t) &= -\xi \omega \sin \omega t; \\
v(x,t) &= -\xi \omega \cos \omega t,
\end{align*}
\]

where \( \eta \) is the surface elevation, positive above the equilibrium level and \( \omega = \sqrt{2g \xi_0 / L} \) is the frequency of the rotation around the center of the basin. The magnitude of the velocity vectors is constant over time at the value \( |V| = \xi \omega \), whereas the direction rotates over time describing an angle \( \alpha = 3 \pi / 2 - \omega t \) during a period \( 0 \leq t \leq T \). This test is extremely severe for numerical models since a great number of cells is continuously wetted and dried.

The test was performed in a square domain \([-1.6 \leq (x,y) \leq +1.6]\) with \( z_0 = 0.05 \) m, \( L = 1 \) m and \( \xi = 0.5 \) m; the basin dimensions are such that the water never reaches the boundaries. The numerical simulation was carried out for four periods with \( \Delta x = \Delta y = 0.02 \) m and \( \Delta t = 3 \times 10^{-6} \) m. Since the water body remains compact and intersects sharply the bathymetry, the numerical solution is not particularly sensitive to the threshold value \( h \).

Fig. 5b shows a contour map of the computed results at \( t = (15/8)T \), when \( x = -\pi/4 \); the shoreline is still circular, the surface almost perfectly planar and the velocity field nearly uniform.

Fig. 6a shows the comparison between numerical and analytical water depth time series in the last two periods of simulation at points \((1.0;0.0), (0.5;0.0), (0.4;0.0)\). The first point gets wet and dry during the periodic motion, the second gets dry only at \( t = (0.5 + n)T, (n \in \mathbb{N}) \), whereas the third remains wet all the time.
An overall quantitative information about the accuracy of the numerical reconstruction is given by $L_2$ error norms for water depth $h$, velocity magnitude $|V|$ and direction $\alpha$ estimated as follows:

\[
L_2(h) = \frac{1}{\bar{h}_{an}(t)} \sqrt{\frac{\sum_{i=1}^{N_{wet}} (h_{i,\text{num}} - h_{i,\text{an}})^2}{N_{wet}}}
\]

\[
L_2(|V|) = \frac{1}{|V|_{an}} \sqrt{\frac{\sum_{i=1}^{N_{wet}} (|V|_{i,\text{num}} - |V|_{i,an})^2}{N_{wet}}}
\]

\[
L_2(\alpha) = \frac{1}{\pi} \sqrt{\frac{\sum_{i=1}^{N_{wet}} (\alpha_{i,\text{num}} - \alpha_{an})^2}{N_{wet}}}
\]

where $N_{wet}$ is the total number of wet cells and $\bar{h}_{an}(t)$ is the average water depth, in this case not depending on time and equal to $z_0/2$.

The trend of the norms during the last two simulation periods is shown in Fig. 6b. The error on the velocity norm and on the direction $\alpha$ are mainly due to the cells close to the shoreline: here water depths become very small and the derived variables $u$ and $v$, obtained by dividing unit discharges by water depth, can assume slightly incorrect values. At the end of four simulation periods, the volume error (relative to the volume at $t=0$) is about $5 \times 10^{-2}$. This residual error is due to a few cells on the shoreline in which water depth remains less than $h$, after the iterative procedure; in these cells water depth is then set at zero. If a pure SGM
reconstruction is performed, spurious oscillations caused by unreliable extrapolations at wet-dry interfaces grow in time and the simulation crashes.

3.3.2. Curved water surface

The second exact solution analyzed concerns the periodic motion of a circular paraboloidal water volume initially at rest and subject to gravity (Fig. 7a). The equations of the motion are [29]:

\[
\begin{align*}
\eta(x, y, t) &= z_0 \left( \sqrt{1 - \left( \frac{x}{A \cos \omega t} \right)^2} - 1 \right) \\
u(x, y, t) &= \frac{1}{A \cos \omega t} \left( \frac{1}{2} \omega x A \sin \omega t \right) \\
v(x, y, t) &= \frac{1}{A \cos \omega t} \left( \frac{1}{2} \omega y A \sin \omega t \right)
\end{align*}
\]

where the frequency \( \omega \) and the parameter \( A \) are:

\[
\omega = \sqrt{\frac{8g}{L}}, \quad A = \frac{(z_0 + \eta_0)^2 - (z_0)^2}{(z_0 + \eta_0)^2 + (z_0)^2}
\]

According to the first of (33), the surface remains a parabola of revolution during the motion and the water body moves away from the center in the first half of the period \( T \) and then converges towards in the second half. Moreover, at times:

\[
t' = nT + \frac{1}{\omega} \arccos \left( \frac{1 - \sqrt{1 - A^2}}{A} \right), \quad n \in \mathbb{N}
\]

![Fig. 7. (a) Definition sketch for Thacker test with curved water surface [29] and (b) slices of numerical results and analytical solution at selected times.](image)

![Fig. 8. Thacker test with curved water surface: comparison between analytical and numerical results for (a) water depth \( h \) and (b) velocity component \( u \). (c) \( L_2 \) error norms for water depth \( h \) and velocity magnitude |\( V \)|.](image)
all the terms between curly brackets in (33) vanish, so the paraboloidal water surface degenerates in a planar surface with $\eta(x,y,t') = 0$.

Due to the large variation over time of the number of wet cells, this problem is extremely severe for numerical models, too. The test was performed in a square domain $[-1.75 \text{ m} \leq (x,y) \leq 1.75 \text{ m}]$ with $z_0 = 0.05 \text{ m}$, $L = 1 \text{ m}$ and $h_0 = 0.10 \text{ m}$; with the assumed parameters the radius of the shoreline at the maximum expansion ($t = (0.5 + n)T, n \in \mathbb{N}$) is $R_{\text{max}} = \sqrt{3} \text{ m}$ and the boundaries of the domain are never reached by the water body. The numerical simulation was carried out for four periods with $\Delta x = \Delta y = 0.02 \text{ m}$ and $h = 5 \times 10^{-6} \text{ m}$.

Fig. 7b shows slices along the $x$-axis of numerical results and analytical solution at some selected times. A moderate underestimation of the water depth occurs near the center after four periods.

Fig. 8a and b show the comparison between numerical and analytical solution for water depth $h$ and velocity component $u$ at points $(0.0; 0.0), (0.5; 0.0), (0.7; 0.7)$. The first two points remain wet all the time, whereas the third gets wet and dry during the periodic motion. Small local differences with the analytical solution are shown more clearly in the insets, where all the computational points are reported.

The $L_2$ norm of $h$ (Fig. 8c) was evaluated according to (32), where the average water depth $h_{\text{an}}(t)$ is equal to $(h(0,0,t) - z_0)/2$. Since the velocity field vanishes at $t = n/2T, n \in \mathbb{N}$, $L_2(|V|)$ was normalized by the analytical velocity magnitude at the shoreline at $t = t'$.

Although the mass error (with respect to the mass at $t = 0$) slightly increases over time for the same reason explained in Section 3.3.1, after four simulation periods the introduction of the flux
correction procedure yields to obtain a relative mass error of about $6 \times 10^{-8}$ without corrupting the tracking of wet/dry fronts; if the flux correction is not applied the error is about 0.01. Like in the previous case, if pure SGM extrapolations are performed, the numerical simulation crashes due to spurious oscillations close to the shoreline.

3.4. Circular dam-break on a non-flat bottom

This test was developed by the authors themselves in order to study the flow consequent to the sudden collapse of an idealized circular dam placed on a non-horizontal bottom. A cylindrical water volume of radius $R_0 = 10$ m is initially placed in a circular domain of radius $R = 25$ m centered in $(x = 0, y = 0)$. The bottom profile is described by the following equation:

$$z(r) = 0.5\left[1 + \cos\left(\frac{2\pi}{5}r\right)\right], \quad (36)$$

where $r = \sqrt{x^2 + y^2}$ is the radius. As initial conditions it is assumed:

$$\begin{align*}
\eta(r, 0) &= 10 \text{ m} \\ h(r, 0) &= 0 \text{ m} \\ u_r(r, 0) &= 0 \text{ m/s}
\end{align*} \quad (37)$$

At the circular boundary a reflective condition is imposed.

Since the problem has a cylindrical symmetry, an inhomogeneous set of 1D differential equations can be derived along $r$ [31]:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial r} = \mathbf{S}_r(\mathbf{U}) + \mathbf{S}_0(\mathbf{U}), \quad (38)$$

where

$$\mathbf{U} = [h, u_r, h]^T, \quad \mathbf{F}(\mathbf{U}) = \left[h u_r, h u_r^2 + \frac{1}{2}gh^2\right]^T, \quad \mathbf{S}_r(\mathbf{U}) = -h u_r \left[0, \frac{u_r}{r}\right]^T, \quad \mathbf{S}_0(\mathbf{U}) = \frac{1}{2}h u_r^2 \left[0, \frac{u_r}{r}\right]^T. \quad (39)$$

In (39) $u_r$ is the radial velocity, $\mathbf{S}_r(\mathbf{U})$ is the source term induced by the metrics and $\mathbf{S}_0(\mathbf{U})$ is the source term due to the non-flat bottom.

A reliable approximation of the exact solution was obtained numerically solving the 1D problem (38) on a very fine mesh ($\Delta r = 0.005$ m). The reference solution was computed by means of the same WSDGM in which the unsplit centered discretization of $\mathbf{S}_0$ allows the satisfaction of the $C$-property, while the splitting of $\mathbf{S}_r$, discretized with a pointwise approach, completes the updating of the conserved variables. In the 2D simulation the computational domain was discretized through a square grid of 0.25 m; the wet/dry tolerance $h_\text{nd}$ was set at $10^{-4}$ m.

Fig. 9 shows 2D numerical profiles of water level and velocity along $\theta = 0$ ($y = 0$) and $\theta = \frac{\pi}{4}$ ($y = x$) at some selected times compared to the 1D profiles computed on the finer mesh. At $t = 0.6$ s the capability of WSDGM to preserve the static condition can be appreciated. At $t = 2$ s and $t = 6$ s the shock wave coming from the

![Fig. 10. Circular dam-break on a non-flat bottom: contour map of numerical water surface level at $t = 2$ s.](image)

![Fig. 11. Circular dam-break on a non-flat bottom: comparison between 1D radial reference solution and 2D numerical profiles of water surface level and velocity at $t = 0.6$ s applying a pure DGM reconstruction.](image)

![Fig. 12. Real field case study: contour map of the area under investigation.](image)
boundary is moving toward the center, whereas at $t = 8$ s the same shock has already passed through the focusing point at $r = 0$ and it is now expanding outwards.

The 2D numerical solution is able to reproduce the complex structures induced by the non-flat bottom and by the reflection against the lateral circular wall. The radial symmetry is well maintained as can be appreciated also from Fig. 10, which shows the contour map of the numerical water surface level at $t = 2$ s.

In Fig. 11 the numerical results obtained adopting $F_{\text{lim}} = 0$ confirms that a pure DGM reconstruction can not maintain the static condition of flow at rest if a centered discretization for the bed slope source term is applied.

Adopting a pure SGM reconstruction ($F_{\text{lim}} \rightarrow \infty$), the scheme does not work at all due to bad extrapolations at wet/dry interfaces.

4. Case study: collapse of the dam on Parma river

In order to test the applicability of WSDGM to real case studies characterized by an irregular topography, the flooding after the

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![Fig. 13. (a) Water stage with velocity vectors and (b) velocity magnitude contour maps at 2 minutes after the breaking.](image1)

![Fig. 14. Water stage contour maps at different times after the breaking.](image2)
hypothetical collapse of the barrage on Parma river (Northern Italy) is presented. The dam, built in 2005 for flood control, is located about 10 km upstream from the city of Parma. The reservoir has a storage capacity of about $12 \times 10^6$ m$^3$ and the maximum water depth with reference to the bottom of the stilling basin is 16.4 m.

The 10 m DTM of the whole area of interest was reconstructed on a through an interpolation of data acquired from the available cartography. Fig. 12 illustrates the contour map of the area together with the different conditions imposed at the boundaries. The reflective condition in the North Eastern limit simulates the presence of a road embankment; it was verified a posteriori that the condition to impose on the western and southern boundaries is not important, since the flooding does not reach these areas. At the initial time, the static condition corresponding to the maximum retaining depth (105.6 ± 1 s.l.) was imposed in the reservoir. The Manning roughness coefficient was set at 0.3 m$^{-1/3}$ in the river and 0.05 m$^{-1/3}$ elsewhere; the wet/dry tolerance $h_c$ was set at $2 \times 10^{-3}$ m. As it can be seen from the flow field (Fig. 13a) and from the contour map of the velocity magnitude (Fig. 13b) at 2 minutes after the breaking, WSDGM is able to preserve the static condition in the region not yet reached by the rarefaction wave. In Fig. 14 water stage contour maps resulting from the numerical modeling at two selected times are shown as an example.

5. Conclusions

In this paper a finite volume MUSCL-Hancock scheme for the numerical integration of 2D shallow water equations was presented. In the framework of the SLIC scheme, the weighted surface-depth gradient method (WSDGM) computes water depth at the cell boundaries through a weighted average, based on the local Froude number, of the extrapolated values deriving from DGM and SGM reconstructions. In particular the scheme applies a pure SGM reconstruction in static conditions when $Fr = 0$, and a pure DGM reconstruction when $Fr$ is greater than an upper limit $Fr_{lim}$, which was set at 2 after a numerical analysis. The WSDGM reconstruction enables the scheme to perform a robust tracking of wet/dry fronts and, together with an unsplit centered discretization of the bed slope source term, to maintain the static condition on non-flat topographies (C-property).

A flux correction applied to the shoreline cells with water depth lower than a threshold value drastically reduces the mass error without corrupting the wet/dry front tracking. WSDGM was validated through its application to a set of severe reference tests involving high bed slopes, 1D and 2D wet/dry fronts. The comparison between reference and numerical results proved that WSDGM provides more accurate solutions than those obtainable applying pure SGM or DGM reconstructions. Moreover, the case study concerning the flooding after the hypothetical collapse of the dam on Parma river confirmed the applicability of WSDGM also for field-scale applications.

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References