Undular bores and secondary waves – Experiments and hybrid finite-volume modelling

Ondes secondaires de Favre – Mesures expérimentales et modélisation par volumes finis hybrides

SANDRA SOARES FRAZÃO, research fellow Fonds National de la Recherche Scientifique, and YVES ZECH, professor, Department of Civil Engineering, Université catholique de Louvain, B 1348 Louvain-la-Neuve, Belgium

ABSTRACT
Secondary free-surface undulations (Favre waves), appearing for example after the opening of a sluice gate or at the head of a bore, cannot be reproduced by numerical models based on the hydrostatic pressure assumption. The Boussinesq equations take into account the extra pressure gradients but are difficult to integrate due to the high-order derivative terms. The paper describes the physics of wave initiation and proposes a demonstration of the Boussinesq equation based on relatively wider assumptions than usually adopted. A linear stability analysis is developed in finite-difference frame to highlight some potential source of numerical instabilities. These conclusions are transposed in a new hybrid finite-volume / finite-difference scheme, which reveals a better accuracy in period and amplitude when evaluated against experiments.

RÉSUMÉ
Les ondes secondaires de surface (ondes de Favre) apparaissant par exemple à l’ouverture d’une vanne de fond ou associées à la tête d’un ressaut mobile, ne peuvent être représentées par des modèles numériques basés sur l’hypothèse d’une distribution hydrostatique de pression. Les équations de Boussinesq tiennent compte des pressions non hydrostatiques mais elles sont difficiles à intégrer avec précision à cause des termes contenant des dérivées partielles d’ordre supérieur. L’article décrit physiquement la naissance des ondes de Favre et propose une démonstration des équations de Boussinesq fondée sur des hypothèses plus larges que celles qui sont généralement utilisées. Une analyse de stabilité linéaire, développée dans le cadre de différences finies, a permis de mettre en évidence quelques sources potentielles d’instabilités numériques. Ces conclusions sont transposées à un schéma nouveau utilisant de manière hybride les volumes finis et les différences finies. Cette méthode, confrontée à des résultats expérimentaux, se révèle plus précise, tant pour la période que pour l’amplitude des ondes.

1. Introduction
The Saint-Venant shallow-water equations, widely used for modelling free-surface flows, rely on an important assumption: as the vertical velocities are assumed to be negligible, the resulting pressure distribution is hydrostatic. This representation has proven to be of satisfying accuracy for representing a wide range of situations like flood flows, or even dam-break induced waves. However, some features like secondary free-surface undulations cannot be reproduced by numerical models based on the hydrostatic-pressure assumption.

In nature, such undular wave trains can be observed for example after the opening of a sluice gate. Depending on the flow conditions, waves with amplitude up to twice the initial bore amplitude can be generated. Those waves were investigated experimentally for the first time by Favre [4] in a rectangular channel. The problems addressed by Favre, and which led him to his important experimental work are linked to hydropower plants and navigation locks. Rapid operation of those systems can induce waves with heights becoming so important that they can damage the channel and river banks. His experimental set-up allowed Favre to measure the development of a wave train induced by a rapid opening or closing of a gate.

More recent experimental work of the same kind has been performed by Treske [17] in channels of rectangular and trapezoidal cross-sections, and by Marche et al. [10] in the dam-break frame-work. The description of Favre waves in a depth-averaged mathematical model requires an extension of the shallow-water equations, such as proposed by Boussinesq [2]. By adopting certain realistic assumptions on the distribution of vertical and horizontal velocity components, extra-terms can be added to the momentum equation, accounting for the changes in the pressure distribution. More recent developments also contributed to the understanding of those weakly non-hydrostatic wave propagation phenomena (Peregrine [12], Schröter [15], Steffler and Jin [16], Prüser and Zielke [13], Marche et al. [10], Nadiga et al. [11]) and are still based on the outstanding initial theoretical work by Boussinesq. Several finite-difference schemes have been developed to solve Boussinesq equations. Peregrine [12] derived such a finite-difference scheme to compute the development of an undular bore, starting from a smooth initial water profile. However, it must be stated that the proposed scheme is unable to compute waves with a steep initial profile (discontinuous free surface).

More recently, Schröter [15] developed a finite-difference scheme for the Boussinesq equations written in variables \((h, q)\) instead of \((h, U)\), \(h\) being the water depth, \(q\) the unit discharge and \(U\) the mean velocity. To ensure the stability of the scheme, the Courant-Friedrichs-Lewy (CFL) number is limited to 0.7, and an additional numerical filter (spline smoothing of computed results) is introduced. This scheme was used by Prüser and Zielke [13] and checked against measurements by Treske [17]. Although good
agreement was obtained for the wave height, significant discrepancies appeared in the wave celerity.

A study on Favre secondary waves associated to a dam-break-induced bore was carried out by Marche et al. [10]. Those authors compared their experimental measurements with computations by a finite-difference model with artificial viscosity (Gharangik and Chaudhry [5]).

All those numerical models present the disadvantages to be either unstable for initially steep wave profiles, or to need the explicit introduction of artificial viscosity to avoid spurious oscillations and instabilities. An attempt is made here to propose a stable finite-volume scheme to compute secondary oscillations even arising from a steep front.

New experimental work has been carried out and is also described in this paper. The results were compared to those obtained by Treske and Favre and showed a good coherence. In consequence, they were used to validate the proposed numerical scheme.

2. Physics of secondary wave initiation

Favre waves, investigated in the laboratory of the Civil Engineering Department at the Université catholique de Louvain, were generated in two different ways. One way was to induce a sudden increase of the discharge in the channel, by raising up rapidly a gate separating two regions of initially constant water depths. This is similar to the experiments carried out by Favre himself, where secondary undulations immediately grow on the initially steep front. Another possibility was to increase the discharge in a flume initially at rest by starting up a pump. The latter method cannot lead to an abrupt rise of the discharge and to a steep initial water profile, as the discharge increase is limited by the inertia of the pump. In this case, what is observed is a rather smooth increase in water level, leading to a wave profile like that illustrated in figure 1. This initially smooth positive wave will steepen while travelling in the downstream direction, and undulations slowly grow at the front head. The growth of these undulations, occurring in both cases of sudden and progressive increase in discharge, is linked to the departure from hydrostatic pressure distribution near the bore front, and is best explained when starting from an initially smooth wave profile. In that case, the assumptions of small curvature of fluid filaments and hydrostatic pressure distribution hold at least until the wave has reached a state where the steepness is such that curvature effects can no longer be neglected.

Peregrine [12] shows how the wave steepening leads to local extra pressure gradients due to the vertical acceleration of the water. However, some aspects of his description appear unsatisfactory, and an attempt to clarify the situation is made here. Consider a long wave in shallow-water like that in figure 1, resulting for example of a progressive increase in discharge. The initial phase of the motion is well described by the Saint-Venant equations, as the change in water level is smooth enough and the pressure in the water is effectively close to a hydrostatic distribution. When travelling, the wave steepens because its local celerity depends on the water depth in such a way that the top of the wave (point A) moves faster than the toe (point E). The water surface curvature becomes then sufficiently important to affect the pressure distribution significantly. Suppose the wave in figure 1 has reached this state.

Points B and D are the points of maximum water-surface curvature, and thus of maximum divergence with the hydrostatic pressure distribution. Curvature at B is such that the pressure there is less than hydrostatic, while at D it is greater than hydrostatic. Those alterations of vertical pressure distribution will induce horizontal pressure gradients as sketched in figure 2. Those, in turn, will generate additional horizontal currents, which, by continuity, will result in vertical displacements of the water surface. The free surface will be raised at B and lowered at D. Points E and A will also undergo the same phenomenon, but in a less significant way. This process will continue and a sequence of waves is formed, which will grow in amplitude until reaching an equilibrium, or breaking.

3. Boussinesq equations

The classical shallow-water depth-averaged equations are obtained on the basis of the hydrostatic pressure assumption. However, the process of vertical integration of the three-dimensional equations of hydrodynamics is by no means limited to nearly horizontal flows and can be extended to flows where some vertical acceleration is allowed. One of the simplest and most widely ap-
Applicable extensions is the one proposed by Boussinesq where the inclination of the fluid filaments is supposed to increase linearly from zero at the bed to a maximum at the free surface.

The demonstration we propose here slightly differs from the classical textbooks (Ligget [9], Peregrine [12]) in the sense that irrotational-flow assumption is not really required for deriving Boussinesq depth-averaged equations. These are obtained by integration of the continuity and movement equations of inviscid fluid. Assuming two-dimensional flow (wide rectangular channel) these equations read

\[\frac{\partial u}{\partial t} + \frac{\partial w}{\partial z} = 0\]  
\[\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} - g\]  
\[\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g\]

To highlight the influence of the vertical velocity component it is convenient to define \( \varphi = w/u \) as the inclination of the fluid filaments regarding the bed, i.e. a kind of measure of the non-parallelism of the streamlines. Making use of the continuity equation (1), the momentum equations (2) and (3) can be rewritten as

\[-\frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{\partial u}{\partial t} - u^2 \frac{\partial (u^2)}{\partial z} = \frac{\partial (u \varphi)}{\partial t} - u \frac{\partial \varphi}{\partial z}\]

The depth-averaged continuity equation is obtained by integrating (1) over the total water depth \( h \). Using Leibniz’s integration rule, and making use of the bottom boundary condition \( w_0 = 0 \) and of the free-surface kinematic condition \( w_0 = \partial h/\partial t + u \partial h/\partial x \) the integration of (1) from \( z = 0 \) to \( z = h \) yields the classical form

\[\frac{\partial h}{\partial t} + \frac{\partial (Uh)}{\partial x} = 0\]  
with the depth-averaged velocity \( U \) defined by \( U h = \int_0^h u \, dz \)

The following assumptions are needed to integrate (4) and (5): (1) the streamline inclination \( \varphi \) increases linearly from zero at the bed to a maximum \( \varphi \) at the free surface; (2) the local velocity may be written \( u = U + u' \) where \( U \) is the depth-averaged velocity and \( u' \) the deviation with depth from this mean value, supposed small compared to \( U \); (3) the velocity deviation \( u' \), the surface inclination \( \varphi \), and the various partial derivatives \( \partial \varphi / \partial t \) (progressive evolution) and \( \partial \varphi / \partial x \) (long waves) are small enough to consider that a product of at least two of these functions is negligible compared to only one of these. Applied to the water surface (kinematic condition), these assumptions yield

\[w_s = u, \varphi_s = \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} \Rightarrow U \varphi_s \approx \frac{\partial h}{\partial t} + U \frac{\partial h}{\partial x}\]  
in such a way that (6) is rewritten, taking into account the linear variation of \( \varphi \)

\[\frac{\partial h}{\partial t} + \frac{\partial (Uh)}{\partial x} = \frac{\partial h}{\partial t} + U \frac{\partial h}{\partial x} + h \frac{\partial U}{\partial x} = U \varphi_s + \]

\[h \frac{\partial U}{\partial x} = h \left( \frac{\partial \varphi}{\partial z} + \frac{\partial U}{\partial x} \right) = 0\]

Integrating (8) from \( z = 0 \) up to \( z \) and making use of the continuity equation (6) yields

\[\varphi = -\frac{z}{h} \frac{\partial U}{\partial x} = \frac{z}{h} \left( \frac{\partial \varphi}{\partial z} + \frac{\partial U}{\partial x} \right)\]

Let us now consider the momentum equations (4) and (5). Integrating (5) from \( z \) to the free surface \( z = h \) where \( p = p_s \) gives an expression for the pressure distribution with an additional term complementing the usual hydrostatic distribution:

\[p = p_s + \rho g (h - z) + \frac{h^2}{2} \left( \frac{\partial h}{\partial t} + U \frac{\partial h}{\partial x} + h \frac{\partial h}{\partial t} + 2U \frac{\partial h}{\partial x} + U^2 \frac{\partial h}{\partial x^2} \right)\]

We can now integrate the equation of motion (4) from \( z = 0 \) to \( z = h \), replacing the pressure \( p \) by (10) and considering that the depth-integration of the velocity deviation \( u' \) is zero:

\[\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + g \frac{\partial h}{\partial x} + \frac{h^2}{2} \left( \frac{\partial h}{\partial t} + U \frac{\partial h}{\partial x} + h \frac{\partial h}{\partial t} + 2U \frac{\partial h}{\partial x} + U^2 \frac{\partial h}{\partial x^2} \right) = 0\]

Equation (11) is similar to the expression proposed by Boussinesq in 1877 and differs from the Saint-Venant momentum equation by the third-order derivative terms. Multiplying (11) by \( h \), adding the depth-averaged continuity equation (6) multiplied by \( U \), and making use of the continuity equation (6) to transform some of the terms in brackets yields

\[\frac{\partial (Uh)}{\partial t} + \frac{\partial (Uh)}{\partial x} \left( U^2 + 2U \frac{\partial h}{\partial x} + h \frac{\partial h}{\partial t} \right) = \frac{h^2}{3} \left( \frac{\partial (Uh)}{\partial t} - U \frac{\partial h}{\partial x} \right) - U \frac{\partial h}{\partial t} - U \frac{\partial h}{\partial x} + U \frac{\partial h}{\partial x} \]

The latter form of the momentum equation is called ‘quasi-conservative’, as the left-hand side is exactly the classical homogeneous and conservative Saint-Venant equation. From this point of view, the non-hydrostatic terms (expression in brackets) on the right-hand side of (12) appear as source terms.

4. Numerical model

Solving the Boussinesq equations (6) and (12) requires a somewhat different scheme than the classical finite-volume integration of the Saint-Venant shallow-water equations. The motion equa-
tion (12) involves third-order derivatives in time and space leading on the one hand to a relaxation of the time-step CFL condition due to the dispersive character of (12), and on the other hand to the unfeasibility of a pure finite-volume scheme which requires a fully conservative form of all the partial derivatives. The dispersive character of (12) can be demonstrated in the case of small-amplitude waves. The long-wave assumption implies that the wave amplitude \( \eta \) is small compared to the mean depth \( H \). Moreover, we only consider shallow-water cases where the water depth is small compared to the wavelength \( H \ll \lambda \). This latter assumption may be proved as equivalent to the condition \( U_0 \partial H / \partial t < \partial H / \partial t \) which implies that the water velocity is small compared to the wave speed (see e.g. Lamb [8]). The continuity equation (6) then reads

\[
\frac{\partial h}{\partial t} + U \frac{\partial h}{\partial x} + \frac{\partial U}{\partial x} = \frac{\partial (H + \eta)}{\partial t} + \frac{\partial H}{\partial t} + \frac{\partial U}{\partial x} = 0
\]  

(14)

The motion equation (11) can be handled similarly after simplifying the terms in brackets in the same way as in (12):

\[
\frac{\partial U}{\partial t} + \frac{U \partial U}{\partial x} + g \frac{h}{\partial x} = \frac{\partial}{\partial t} \left[ \frac{h^2}{3} \left( \frac{\partial^2 U}{\partial t \partial x^2} + \frac{\partial U}{\partial x} \right) \right] + \frac{\partial h}{\partial t} + \frac{h}{\partial t} \frac{\partial U}{\partial x} \]

\[
= \frac{\partial U}{\partial t} + g \frac{\partial h}{\partial x} + \frac{\partial (H + \eta)}{\partial t} + \frac{H}{\partial t} \frac{\partial U}{\partial x} + \frac{h}{\partial t} \frac{\partial U}{\partial x}
\]

\[
\frac{\partial U}{\partial t} + g \frac{\partial h}{\partial x} + \psi H^2 \frac{\partial^3 U}{\partial t \partial x^2} = 0
\]  

(15)

where \( \psi \) is a constant equal to unity introduced to ‘mark’ the non-hydrostatic correction. The system (14)-(15) admits a plane-wave solution of the type

\[
\begin{pmatrix}
\eta \\
U
\end{pmatrix} = Re \begin{pmatrix}
A \\
B
\end{pmatrix} e^{i(\omega t - kx)}
\]

consisting in one term of a Fourier-series expansion of an oscillatory wave. In (16), \( Re \) is the ‘real part’ operator, \( A \) and \( B \) are complex constants, \( \lambda = cT \) the wavelength, \( c \) the wave celerity, \( T \) the period, \( k = 2\pi/\lambda \) is the wave number and \( \varphi = 2\pi c/\lambda = 2\pi T \) the angular frequency.

Substituting (16) into (14) and (15) yields a homogenous system of algebraic equations

\[
\begin{pmatrix}
-i \omega \\
 k H \\
- i \omega \left(1 + \frac{\psi}{3} k^2 H^2\right)
\end{pmatrix} \begin{pmatrix}
A \\
B
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

(17)

which has a solution only if its coefficient matrix is singular. This latter condition gives the dispersion relation

\[
\omega^2 \left(1 + \frac{\psi}{3} k^2 H^2\right) = k^2 g H
\]

(18)

i.e. a relation giving \( \omega \) as a function of \( k \), leading to the following expression of the wave celerity

\[
c = \frac{\omega}{k} = \sqrt{g H / \left[1 + \frac{\psi}{3} k^2 H^2\right]}
\]

(19)

If the celerity is a function of the wave number \( k \) and thus of the wavelength \( \lambda \), the waves are called ‘dispersive’, because waves of different lengths, propagating at different speed, ‘disperse’, or separate (Kundu [7]). We can see that the non-hydrostatic term here is responsible for the dispersive character of the waves. Indeed, neglecting this term (\( \psi = 0 \)), we recover the shallow-water equations in which the wave speed or celerity \( c = \sqrt{g H} \) is independent of the wavelength. This dispersive nature of the waves challenges the numerical scheme to reproduce accurately the wave periods.

4.2. Linear stability analysis for finite-difference scheme

In a space-time grid where \( n \) is the time \((n \Delta t)\) and \( j \) the location \((j \Delta x)\), equations (14) and (15) read in finite-difference form

\[
\begin{pmatrix}
\eta^r_j \eta^i_j \\
U^r_j U^i_j
\end{pmatrix} = \frac{\eta^{r+1}_j - \eta^i_j}{\Delta t} + \frac{U^{r+1}_j - U^i_j}{2 \Delta x} = 0
\]

(20)

\[
\begin{pmatrix}
U^r_j U^i_j
\end{pmatrix} = \frac{\eta^{r+1}_j - \eta^{r+1}_j}{\Delta t} = \frac{U^{r+1}_j - U^r_j}{2 \Delta x}
\]

(21)

\[
\begin{pmatrix}
U^r_j U^i_j
\end{pmatrix} = \frac{\eta^{r+1}_j - \eta^{r+1}_j}{\Delta t} = \frac{U^{r+1}_j - U^r_j}{2 \Delta x}
\]

\[
\begin{pmatrix}
U^r_j U^i_j
\end{pmatrix} = \frac{\eta^{r+1}_j - \eta^{r+1}_j}{\Delta t} = \frac{U^{r+1}_j - U^r_j}{2 \Delta x}
\]

\[
\begin{pmatrix}
U^r_j U^i_j
\end{pmatrix} = \frac{\eta^{r+1}_j - \eta^{r+1}_j}{\Delta t} = \frac{U^{r+1}_j - U^r_j}{2 \Delta x}
\]

\[
\begin{pmatrix}
U^r_j U^i_j
\end{pmatrix} = \frac{\eta^{r+1}_j - \eta^{r+1}_j}{\Delta t} = \frac{U^{r+1}_j - U^r_j}{2 \Delta x}
\]

\[
\begin{pmatrix}
U^r_j U^i_j
\end{pmatrix} = \frac{\eta^{r+1}_j - \eta^{r+1}_j}{\Delta t} = \frac{U^{r+1}_j - U^r_j}{2 \Delta x}
\]

\[
\begin{pmatrix}
U^r_j U^i_j
\end{pmatrix} = \frac{\eta^{r+1}_j - \eta^{r+1}_j}{\Delta t} = \frac{U^{r+1}_j - U^r_j}{2 \Delta x}
\]
Note that the second term of (21) \( g \frac{\partial \eta}{\partial x} \) is defined at time \( n+1 \), which is required for the stability of the numerical scheme. Assuming a plane-wave solution

\[
\begin{align*}
\left( \eta_i^n, U_i^n \right) &= Re \left[ \left( \begin{array}{c}
A \\
B
\end{array} \right) e^{i(k \Delta x - \omega \Delta t)} \right] = Re \left[ \left( \begin{array}{c}
A \\
B
\end{array} \right) \left( \alpha \right) e^{i(\theta)} \right]
\end{align*}
\]

(22)

with \( \theta = k \Delta x \) and where \( \alpha = e^{-i \Delta \theta} \) is the time-dependent part of the solution. The system (20)-(21) can be rewritten as

\[
\begin{bmatrix}
\frac{\alpha - 1}{\Delta t} \\
\frac{e^{i \theta} - e^{-i \theta}}{2 \Delta x} \frac{H e^{i \theta} - e^{-i \theta}}{2 \Delta x} \\
\frac{\alpha - 1}{\Delta t} \left( 1 - \frac{\psi H^2}{3 \Delta x^2} \right) \left( e^{i \theta} - 2 + e^{-i \theta} \right)
\end{bmatrix}
\begin{bmatrix}
A \\
B
\end{bmatrix}
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

(23)

Using the relations \( e^{i \theta} \), the dispersion relation reads

\[
\alpha^2 - 2 \beta \alpha + 1 = 0 \quad \text{with} \quad \beta = 1 - \frac{g H}{2} \frac{\sin^2 \theta}{1 + \psi \frac{2 H^2}{3 \Delta x^2} (1 - \cos \theta)} \frac{\Delta t^2}{\Delta x^2}
\]

(24)

The scheme will be stable if the amplitude of the solution (22) does not grow with time, i.e. \( |\alpha| \leq 1 \). The stability condition is fulfilled if \( -1 \leq \beta \leq 1 \), leading to the amplification factor \( |\alpha| = 1 \) and to the following condition on the time step \( \Delta t \)

\[
\Delta t \leq \frac{\Delta x}{\sqrt{gh}} \left( 1 + \frac{\psi \frac{2 H^2}{3 \Delta x^2} (1 - \cos \theta)}{(1 + \psi \frac{2 H^2}{3 \Delta x^2} (1 - \cos \theta))} \right)
\]

(25)

which is less severe than the classical condition prevailing for hydrostatic case with \( \psi = 0 \) (see e.g. Abbott and Basco, [1])

\[
\Delta t \leq \frac{2 \Delta x}{\sqrt{gh}}
\]

(26)

Applying this analysis to a scheme where the second term \( g \frac{\partial \eta}{\partial x} \) of (21) is defined at time \( n \) instead of \( n+1 \), the stability condition would never be fulfilled. This result, strictly ascertained for the present finite-difference scheme, will also prevail for the hybrid finite-volume scheme proposed in the next section.

4.3 Hybrid finite-volume scheme

The idea is to apply a standard finite-volume scheme to the conservative form (6) and (12) of the Boussinesq equations, combined with a finite-difference treatment of the non-hydrostatic term, considered as a source term. Equation (6) and (12) can be rewritten in vector form as

\[
\frac{\partial U_i}{\partial t} + \frac{\partial F(U_i)}{\partial x} = S
\]

with

\[
U_i = \begin{bmatrix} h \\ \mathbf{u} \end{bmatrix}, \quad F(U_i) = \begin{bmatrix} U_i h \\ U_i^2 h + g h^2 \end{bmatrix}
\]

(27)

where \( q = U_i h \) is the mass flux and \( \sigma = U_i^2 h + g h^2 \) is the momentum flux. A finite-volume discretization (Hirsch [6]) of the integral form of (27) applied to a mesh of the space-time computational grid of figure 3 yields

\[
U_i^{n+1} = U_i^n + \frac{\Delta t}{\Delta x} \left( \overline{F}_{i+1/2} - \overline{F}_{i-1/2} \right) + S_i \Delta t
\]

(28)

The inter-cell fluxes \( \overline{F}_{i+1/2} \) and \( \overline{F}_{i-1/2} \) may be calculated as functions of initial vectors \( U_i \) and \( U_{i+1} \) using for example mean values (superscript *) at the interface

\[
\overline{F}_{i+1/2} = \frac{1}{2} \left( F_i + (1-f_i)(F_i^*) \right) + f_i \left( F_i^* \right) (U_i - U_{i+1})
\]

(29)

Fig. 3 Computational grid
where \( f_0(Fr) \) and \( f_1(Fr) \) are respectively upwind and diffusion functions of the Froude number \( Fr \). The well-known Roe scheme [14] uses (29) with the following definitions

\[
Fr^* = \frac{U^*}{c^*}, \quad U^* = \frac{c_i U_i + c_{i+1} U_{i+1}}{c_i + c_{i+1}}, \quad c^* = \sqrt{\frac{g h_i + h_{i+1}}{2}} \tag{30}
\]

\[
f_u(Fr^*) = \frac{1 + Fr^*}{2}, \quad f_d(Fr^*) = \frac{1 - Fr^*}{2}
\]

In order to take into account the stability condition pointed out for finite-difference schemes, a special treatment of the pressure term is required. The following predictor-corrector scheme is thus proposed to solve (28).

**Predictor step for the continuity equation**

The continuity equation in (28) is solved explicitly in a finite-volume scheme to find a provisional value of \( h_i^{n+1} \). Let this value be \( h_i^{(n+1)} \). We obtain

\[
h_i^{(n+1)} = h_i^n + \frac{\Delta t}{\Delta x} (\bar{q}_{i+1/2}^{\text{pred}} - \bar{q}_{i-1/2}^{\text{pred}}) \tag{31}
\]

where \( \bar{q}_{i+1/2}^{\text{pred}} \) is the mass flux at the interface between cells \( i \) and \( i+1 \) calculated using all variables at time \( n \) according to (29)-(30)

\[
\bar{q}_{i+1/2}^{\text{pred}} = \frac{1 + Fr^*}{2} q_i^n + \frac{1 - Fr^*}{2} q_{i+1}^n + \frac{1}{2} \left( Fr^* \right)^2 (h_i^n - h_{i+1}^n) \tag{32}
\]

**Implicit step for the momentum equation**

The momentum equation in (28) reads, using a finite-difference approximation for the source terms

\[
q_i^{n+1} - q_i^n = \frac{\Delta t}{\Delta x} \left( \bar{q}_{i+1/2}^{\text{corr}} - \bar{q}_{i-1/2}^{\text{corr}} \right) + \frac{\Delta t}{\Delta x} \frac{\bar{h}_{i+1/2}^{(n+1)}}{3} \frac{q_{i+1}^{n+1} - q_{i+1}^n}{h_{i+1}^{n+1} - h_{i+1}^n} + \frac{\Delta t}{\Delta x} \frac{q_i^{n+1} - q_i^n}{h_i^{n+1} - h_i^n} + \frac{\Delta t}{\Delta x} \frac{q_{i-1}^{n+1} - q_{i-1}^n}{h_{i-1}^{n+1} - h_{i-1}^n} \tag{33}
\]

This can be rewritten as

\[
- \frac{1}{3} \left( \bar{h}_{i+1/2}^{(n+1)} \right) q_{i+1}^{n+1} + \frac{1 + 2}{3} \left( \bar{h}_{i+1/2}^{(n+1)} \right) q_{i+1}^{n+1} - \frac{1}{3} \left( \bar{h}_{i+1/2}^{(n+1)} \right) q_{i+1}^{n+1} = a_{i+2} q_{i+2}^n + a_{i+1} q_{i+1}^n + a_i q_i^n + a_{i-1} q_{i-1}^n + a_{i-2} q_{i-2}^n + \frac{\Delta t}{\Delta x} \left( \bar{q}_{i+1/2}^{\text{corr}} - \bar{q}_{i-1/2}^{\text{corr}} \right) \tag{34}
\]

The momentum flux \( \bar{q}_{i+1/2}^{\text{corr}} \) is evaluated by Roe’s model, according to (29)-(30), but with the hydrostatic-pressure term calculated with the provisional value \( h_i^{(n+1)} \) from (31), to ensure the stability condition

\[
\bar{q}_{i+1/2}^{\text{corr}} = \frac{1 + Fr^*}{2} \left( \left( U^2 h_i \right)^{n+1} + \left( g h_i^{(n+1)} \right)^{n+1} \right) + \frac{1 - Fr^*}{2} \left( \left( U^2 h_i \right)^{n+1} + \left( g h_i^{(n+1)} \right)^{n+1} \right) + c^* \left( Fr^* \right)^2 (h_i^n - h_{i+1}^n) \tag{35}
\]

where the mean interface values (with superscript * ) are computed from (30) using the variables at time \( n \).

**Corrector step for the continuity equation**

The continuity equation is solved again to find \( h_i^{n+1} \):

\[
h_i^{n+1} = h_i^n + \frac{\Delta t}{\Delta x} (\bar{q}_{i+1/2}^{\text{corr}} - \bar{q}_{i-1/2}^{\text{corr}}) \tag{36}
\]

The mass flux \( \bar{q}_{i+1/2}^{\text{corr}} \) in (36) is evaluated with time-averaged values of the mass fluxes \( q \)

\[
\bar{q}_{i+1/2}^{\text{corr}} = \frac{1 + Fr^*}{2} q_{i+1}^n + \frac{1 - Fr^*}{2} q_i^n + \frac{1 - Fr^*}{2} q_{i+1}^n + \frac{1 + Fr^*}{2} q_i^n - c^* \left( Fr^* \right)^2 (h_i^n - h_{i+1}^n) \tag{37}
\]

This scheme is first-order accurate in space, and it was observed that the intrinsic numerical diffusion significantly damps the effects of the non-hydrostatic source term. To avoid this inconvenience, second-order accuracy in space has been introduced using the MUSCL scheme (Hirsch [6]), consisting in a linear reconstruction of the free-surface which is thus no more composed of cells with constant water depth.

5. Experiments

5.1. Experimental set-up

An experimental flume was built in the laboratory of the Civil Engineering Department at the University of Louvain (UCL). Figure 4 shows the channel configuration. A gate separates two regions of initially constant water depth : the upstream reservoir and the channel itself, both at an initial rest state. Water-level gauges (C0…C5) were placed in the downstream part of the channel, to measure the time evolution of the water level.

When opening the gate rapidly, a bore travels into the downstream channel. The wave Froude number \( Fr \), defined later in equation (41), will determine the bore aspect. In the supercritical case (\( Fr \gg 1 \)), the bore will consist of a steep front. On the other hand, in the near-critical state (\( Fr \approx 1 \)), undulations will grow at
the bore head, leading to a travelling wave train (see figure 5). The picture is taken from the downstream end of the channel. The undulations growing at the front of the wave can be clearly identified. Figure 6 shows the water surface computed with the above-presented numerical model. The qualitative agreement with the picture feature is good. The first wave has the maximum amplitude and is followed by waves of decreasing heights. For higher Froude numbers, the waves will break and the wave train will take the form of a steep bore.

5.2. Comparison with other experiments

In this kind of experiments, the Froude number is defined as the ratio between the wave speed \( a - U_1 \) and the celerity \( c = \sqrt{gH_1} \), where \( a \) is the absolute speed of the discontinuity (see figure 7).

The following continuity and momentum relations can be written across the discontinuity

\[(U_z - a)H_2 = (U_1 - a)H_1\]
\[(U_z - a)H_2 + g \frac{H_2^2}{2} = (U_1 - a)H_1 + g \frac{H_1^2}{2}\]

(38) \hspace{1cm} (39)

giving the relative speed of the discontinuity

\[a - U_1 = \sqrt{\frac{1}{2} g \frac{H_2^2}{H_1} (H_2 + H_1)}\]

(40)

The Froude number can thus be expressed as

\[Fr = \frac{a - U_1}{c} = \sqrt{\frac{H_2}{2H_1} (H_2 + H_1)}\]

(41)

As already mentioned, the Froude number is the important parameter for classifying the waves. Its value will range between 1 and 1.25 … 1.3. For higher Froude numbers, the first wave will break or even the wave train will completely disappear and become a steep travelling bore. Figure 8 shows a comparison of the maximum \( z_{\text{max}} \) (upper point series), the minimum \( z_{\text{min}} \) (lower series) and the mean value \( z_m = H_2 - H_1 \) (intermediate series around the curve corresponding to equation 41) of the wave height for three
Fig. 7 Definition sketch for the Froude number

experimental series: Favre, Treske and the experiments described previously. For all series, the measurements are in good agreement with past investigations. It clearly appears that for a certain limit value of the Froude number, the maximum relative wave height decreases, which denotes the breaking of the wave.

6. Validation of the numerical model

Figure 9 shows computed profiles during the development of an undular bore, starting from an initially smooth wave profile, corresponding to a smooth increase of the discharge in a channel initially at rest. This is equivalent to the situation illustrated in the second section to explain the physics of the growth of undulations in a weakly non-hydrostatic system. The initial wave profile is proposed by Peregrine [12], and is such that the initial motion can be described by a hydrostatic theory. Results computed with the Saint-Venant shallow-water equations are compared to the Boussinesq equations solved by a finite-difference scheme (Peregrine [12]) and by the proposed hybrid finite-volume scheme (see figure 9). It must be outlined that Peregrine’s scheme solves simplified equations where the non-hydrostatic term is linearized, assuming small amplitude and $U \frac{\partial z}{\partial x} \ll \frac{\partial z}{\partial t}$, which is not the case for our scheme.

For the clarity of the figure, the successive profiles are shifted. The steepening of the wave computed with the hydrostatic theory (Saint-Venant equations) clearly appears, while with the Boussinesq equations undulations grow at the head of the bore. No significant differences appear between both non-hydrostatic computed results, indicating that in this case of smooth initial wave profile, the assumptions chosen by Peregrine [12] to linearize the non-hydrostatic term are valid. This is however not always the case as will be shown in the next comparisons between computed results and experimental measurements.

As already mentioned, Peregrine’s finite-difference scheme was unable to reproduce the experiments starting with a steep front. However, with the hybrid finite-volume scheme, a discontinuous initial profile was successfully used in the following computation. Figure 10 shows the measured and computed water depth evolution at the 6 gauging points in the experimental channel. The agreement is good, in particular the amplitude of the first crest and the wave propagation celerity are well represented.

Figure 11 shows close-up comparisons for different Froude numbers. Agreement is good; the wavelengths are well reproduced, with a slight overestimation of the dispersive character of the waves. The main discrepancies with the measured profiles lie in the damping of the wave: the computed undulations vanish quicker than the measured ones. However, figure 11d clearly shows that the complete wave train is well represented, with a good accuracy of the computed $H_2$ water depth. The small perturbations in the measured water profile after $t = 12$ s are due to lo-

---

**Fig. 8** Comparison with experimental series by Favre and Treske (● Favre, □ Treske, Q present study)

**Fig. 9** Development of an undular bore, —— hydrostatic theory, —— Peregrine finite-difference scheme (upper curve) and hybrid finite-volume scheme (lower curve)
cal 2D reflection effects induced by little imperfections in the experimental flume walls.

The good agreement of the wavelengths is not affected by the Froude number. However, although the first wave amplitude is well computed, a trend to underestimate the subsequent wave amplitudes is observed for increasing Froude numbers. This might be considered as an intrinsic limit of the Boussinesq equations, as some essential assumptions made in their derivation are less valid in this case. Indeed, as $\eta_s, \phi_t, \phi_x$ become larger, their products are no longer negligible.

The consequences of linearizing the non-hydrostatic term did not clearly appear on figure 9 when comparing Peregrine’s scheme to the present hybrid finite-volume scheme on a smooth wave case. Figure 12 compares results computed with the hybrid finite-volum

volume scheme, on the one hand with the complete non-hydrostatic term, and on the other hand with this term linearized in the same way as in the equations solved by Peregrine.

The figure shows that linearizing the non-hydrostatic term increases the error on the wavelength, i.e. the dispersive character of the computed wave. This highlights the need to take all terms into account since the assumptions used to linearize the non-hydrostatic term are not really fulfilled here (especially $\eta \ll H$).

Finally, figure 13 shows the influence of the time step, represented here by the CFL number:

$$CFL = \frac{\max[U + c]}{Ax/At} \leq 1$$

(42)

This definition strictly applies for shallow-water explicit schemes. For non-hydrostatic wave propagation, the linear stability analysis conducted in a previous section resulted in the time-step condition (25), taking into account the implicit treatment of the non-hydrostatic terms. The hybrid finite-volume scheme combines both approaches with the consequences that the CFL number has to be smaller than 1 for stability, but also that the numerical accuracy increases for still smaller CFL numbers. So, as illustrated in fig-

![Fig. 10 Water level evolution at the gauging points – Fr = 1.104, experiments, hybrid finite-volume scheme](image1)

![Fig. 11 Water level evolution at some gauging points for different Froude numbers, experiments, hybrid finite-volume scheme](image2)

![Fig. 12 Consequences of linearizing the non-hydrostatic term, experiments, hybrid finite-volume scheme with linearized and non-linearized non-hydrostatic term](image3)

![Fig. 13 Influence of the time step, experiments and hybrid finite-volume scheme](image4)
ure 13, although the computations are stable for any CFL number smaller than 1, the accuracy in amplitude increases when decreasing the time step.

7. Conclusions

A new numerical scheme for computing weak non-hydrostatic flows is presented. This scheme solves the Boussinesq equations by a hybrid finite-volume method, consisting in applying a finite-volume scheme for the conservative part of the equations and finite differences for the non-hydrostatic terms. The propagation of an undular bore, studied for the first time experimentally by Favre, is a typical example of such weak non-hydrostatic flows. It was observed that undulations appear if the Froude number (41) ranges between 1 and 1.28. For higher Froude numbers, the waves break, leading to a steep front. New experiments have been carried out, and appear to compare well with similar series by other authors. The results were used to validate the numerical scheme. In the range of validity of the Boussinesq equations, the computed and measured profiles are in good agreement, although the computed wave length accuracy could be improved. Linearizing the pressure term, as proposed by Peregrine, works well for mild waves. For steep initial conditions, the proposed scheme constitutes an improvement of the existing finite differences schemes in that it is able to accurately predict the wave celerity, height, and wavelength, without any artificial viscosity usually needed to increase the numerical stability.

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Notations

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>absolute speed of the discontinuity</td>
</tr>
<tr>
<td>c</td>
<td>wave celerity</td>
</tr>
<tr>
<td>CFL</td>
<td>Courant number</td>
</tr>
<tr>
<td>F</td>
<td>flux vector</td>
</tr>
<tr>
<td>Fr</td>
<td>Froude number</td>
</tr>
<tr>
<td>g</td>
<td>acceleration due to gravity</td>
</tr>
<tr>
<td>h</td>
<td>water depth</td>
</tr>
<tr>
<td>H</td>
<td>mean water depth (constant)</td>
</tr>
<tr>
<td>k</td>
<td>wave number D/2</td>
</tr>
<tr>
<td>p</td>
<td>pressure</td>
</tr>
<tr>
<td>q</td>
<td>unit discharge and mass flux</td>
</tr>
<tr>
<td>S</td>
<td>vector of source terms</td>
</tr>
<tr>
<td>t</td>
<td>time</td>
</tr>
<tr>
<td>T</td>
<td>wave period</td>
</tr>
<tr>
<td>U</td>
<td>depth-averaged velocity</td>
</tr>
<tr>
<td>v, w</td>
<td>horizontal and vertical velocities</td>
</tr>
<tr>
<td>u'</td>
<td>deviation from the depth-averaged velocity u – U</td>
</tr>
<tr>
<td>x, z</td>
<td>horizontal and vertical co-ordinates</td>
</tr>
<tr>
<td>α</td>
<td>amplification factor</td>
</tr>
<tr>
<td>η</td>
<td>free-surface elevation above H</td>
</tr>
<tr>
<td>θ</td>
<td>phase angle</td>
</tr>
<tr>
<td>λ</td>
<td>wavelength</td>
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<td>ρ</td>
<td>water density</td>
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<td>σ</td>
<td>momentum flux</td>
</tr>
<tr>
<td>φ</td>
<td>streamline inclination</td>
</tr>
<tr>
<td>τ</td>
<td>non-hydrostatic ‘marker’, equal to unity</td>
</tr>
<tr>
<td>ω</td>
<td>wave angular frequency</td>
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</table>

References
